Lower bounds in some power sum problems

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Abstract

We prove that for $j \geq 0$ one has that

$$\inf_{|z_k|=1} \max_{\nu=1,\ldots,n^2+j} \left| \sum_{k=1}^n z_k^\nu \right| \geq \sqrt{n + \frac{(1+j)(n-1)}{2(j+n^2)}}. \tag{*}$$

This improves upon former estimates. Our proof will use Fejér kernels. We also prove corresponding results for non pure power sums, and the pure power sum estimate

$$\left(\sqrt{2 - \frac{2}{\alpha}} - o(1) \right) \sqrt{n} \leq \inf_{|z_k|=1} \max_{\nu=1,\ldots,\lfloor \alpha n^2 \rfloor} \left| \sum_{k=1}^n z_k^\nu \right|,$$

for constants $\alpha > 1$. This improves further on (*) when $j \geq 2n^2$.

1 Introduction

The power sum method of Turán (see Turán [Tur84] or Montgomery [Mon94] Chapter 5) allows us to obtain lower bounds for power sums

$$\max_{\nu=N(n),\ldots,M(n)} |g(\nu)|, \tag{1}$$

where

$$g(\nu) = \sum_{k=1}^n b_k z_k^\nu \tag{2}$$

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for z_k and b_k complex numbers, and $M(n) - N(n) \geq n$ a function of n. We will henceforth assume that the $b_k \geq 0$ are positive. In particular we are interested in the case of *pure* power sums ($b_k = 1$)

$$S(\nu) = \sum_{k=1}^{n} z_k^{\nu}, \tag{3}$$

and the minimum norm $\min_k |z_k| = 1$. We will also assume that $N(n) = 1$. In this case a number of results have been proved.

$$\max_{\nu=1,\ldots,n} |S(\nu)| \geq 1, \qquad \text{(Turán [Tur69])}$$

$$\max_{\nu=1,\ldots,2nm-m(m+1)+1} |S(\nu)| \geq \sqrt{m}. \qquad (1 \leq m \leq n) \qquad \text{(Andersson [And96])}$$

Under the min norm it seems reasonable that the minimal systems $(z_1, \ldots, z_n)$ which minimize the expressions actually lies on, or very close to the unit circle. This has been difficult to prove and in fact in that case, when the z_k are unimodular, Newman, Cassels and Szalay have independently proved the stronger result

$$\max_{1 \leq \nu \leq cn} \left| \sum_{k=1}^{n} z_k^{\nu} \right| \geq \sqrt{\frac{cn - n + 1}{c}}. \qquad \text{([Tur84], Theorem 7.3)}$$

2 One sided bounds

We will denote

$$g(\nu) = \sum_{k=1}^{n} b_k e(\theta_k \nu), \tag{4}$$

where θ_k are real numbers and $b_k > 0$. We will let

$$A = g(0) = \sum_{k=1}^{n} b_k, \qquad \text{and} \qquad B = \sum_{k=1}^{n} b_k^2.$$

In this section we will furthermore assume that g is real valued. In particular this implies that

$$g(\nu) = g(-\nu). \tag{5}$$

We will also let

$$g^+(\nu) = \begin{cases} g(\nu), & g(\nu) > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad g^-(\nu) = \begin{cases} g(\nu), & g(\nu) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that

$$g(\nu) = g^+(\nu) + g^-(\nu),$$

and

$$|g(\nu)| = g^+(\nu) - g^-(\nu). \tag{6}$$

Our method of proof will use the Fejér kernel

$$F_{m+1}(x) = \sum_{\nu=-m}^{m} \left(1 - \frac{|\nu|}{m+1}\right) e(\nu x). \tag{7}$$

The Fejér kernel can be written as

$$F_{m+1}(x) = \frac{1}{m+1} \left(\frac{\sin \pi (m+1)x}{\sin \pi x}\right)^2, \tag{8}$$

and is thus non negative. We will let

$$\alpha = \frac{2}{m} \sum_{\substack{g^+(\nu)>0 \\ 1 \leq \nu \leq m}} \left(1 - \frac{\nu}{m+1}\right), \qquad \text{and} \qquad \beta = \frac{2}{m} \sum_{\substack{g^-(\nu)<0 \\ 1 \leq \nu \leq m}} \left(1 - \frac{\nu}{m+1}\right). \tag{9}$$

From the representation (8) it follows that $F_{m+1}(0) = m+1$. From this it is clear that

$$\sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) = \frac{m}{2}, \tag{10}$$

and thus also

$$\alpha + \beta \leq 1. \tag{11}$$

2.1 The first method

Lemma 1. *One has that*

$$\sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) |g(\nu)|^2 \geq \frac{(m+1)B - A^2}{2}.$$

Proof. We have that

$$\sum_{\nu=-m}^{m} \left(1 - \frac{|\nu|}{m+1}\right) |g(\nu)|^2 = \sum_{k,l=1}^{n} b_k b_l F_{m+1}(\theta_k - \theta_l),$$

which by the contribution of the diagonal $k = l$, and the non negativity of the Fejér kernel, eq. (8) implies that

$$\sum_{\nu=-m}^{m} \left(1 - \frac{|\nu|}{m+1}\right) |g(\nu)|^2 \geq \sum_{k=1}^{n} b_k^2 F_{m+1}(0).$$

The result follows by subtracting the term $\nu = 0$ and using equation (5). $\square$

Lemma 2. *Suppose that $|g(v)| \leq M$ for $\nu = 1, \ldots, m$. Then*

$$\sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) g^+(\nu) \geq \frac{B(m+1) - AM - A^2}{4M}.$$

Proof. Since $g^+(\nu) = (|g(\nu)| + g(\nu))/2$ and obviously $|g(\nu)| \geq |g(\nu)|^2/M$ when $\nu \neq 0$ we have that

$$\sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) g^+(\nu) \geq \frac{1}{2} \sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) \left(\frac{|g(\nu)|^2}{M} + g(\nu)\right).$$

The first term can be estimated by Lemma 1 and gives the contribution

$$\frac{B(m+1) - A^2}{4M}.$$

The second term can be investigated by use of the Fejér kernel. By the non negativity of the Fejér kernel, equation (8) we have that

$$\sum_{\nu=-m}^{m} \left(1 - \frac{|\nu|}{m+1}\right) g(\nu) \geq 0$$

By the fact that $g(0) = A$ and using equation (5) we find that

$$\sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) g(\nu) \geq -\frac{A}{2}, \tag{12}$$

which gives the remaining contribution to our Lemma. $\square$

We now prove the following Theorem.

Theorem 1. *Suppose that $|g(\nu)| \leq M$ for $\nu = 1, \ldots, m$. Then one has that*

$$\max_{\nu=1,\ldots,m} g^+(\nu) \geq \frac{B(m+1) - AM - A^2}{2Mm}.$$

Proof. This follows from Lemma 2 and equation (10). $\square$

2.2 An improvement for large values of m

As m tends to infinity Theorem 1 will give us

$$\max_{\nu=1,\ldots,m} g^+(\nu) \geq \frac{B}{2M} - o(1),$$

We will prove a stronger results which allows us to obtain

$$\max_{\nu=1,\ldots,m} g^+(\nu) \geq M + 2 - \frac{2M^2}{B} - o(1).$$

This will give sharper results when $M \asymp \sqrt{B}$ and for large m.

Theorem 2. *Suppose that $|g(\nu)| \leq M$ for $\nu = 1, \ldots, m$. If $B(m+1) - A^2 - mM^2 \geq 0$ then one has that*

$$\max_{\nu=1,\ldots,m} g^+(\nu) \geq \frac{B(m+1) - A^2}{mM},$$

In case $B(m+1) - A^2 - mM^2 \leq 0$ one has that

$$\max_{\nu=1,\ldots,m} g^+(\nu) \geq M + 2 \times \frac{B(m+1) - A^2 - mM^2}{B(m+1) - A^2 - AM}$$

under the assumption that the denominator in the last fraction is positive.

Proof. By equations (9) and (11) it is clear that

$$\frac{2}{m} \sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) g^-(\nu) \geq -M(1-\alpha). \tag{13}$$

By equation (6) and the fact that $|g(\nu)| \leq M$ we get the inequality

$$g^+(\nu) \geq \frac{|g(\nu)|^2}{M} + g^-(\nu).$$

By combining this with equation (13) we see that

$$\frac{2}{m} \sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) g^+(\nu) \geq \frac{2}{m} \sum_{\nu=1}^{m} \left(1 - \frac{\nu}{m+1}\right) \frac{|g(\nu)|^2}{M} - M(1-\alpha).$$

which by Lemma 1 can be estimated by

$$\geq \frac{B(m+1) - A^2}{Mm} - M(1-\alpha).$$

This together with the definition of α, equation (9) implies that

$$g^+(\nu) \geq \frac{1}{\alpha} \cdot \left(\frac{B(m+1) - A^2}{Mm} - M(1-\alpha) \right) = \frac{B(m+1) - A^2 - mM^2}{\alpha Mm} + M \tag{14}$$

for some $\nu = 1, \dots, m$. We see that if $B(m+1) - A^2 - mM^2 \geq 0$ then the function is decreasing in α and the minimum over $0 < \alpha \leq 1$ is attained for $\alpha = 1$. This gives us case 1. In the case when $B(m+1) - A^2 - mM^2 \leq 0$ the function is increasing in α and we use the following estimate

$$\alpha \geq \frac{B(m+1) - AM - A^2}{2mM}.$$

which follows from Lemma 2 to obtain a lower bound. By putting this value in the right hand side of (14) we obtain a lower bound and we obtain the second part of our theorem. We remark that we also need that $B(m+1) - AM - A^2$ is positive since otherwise we get an $\alpha < 0$. ☐

3 A lower bound for power sums

We will now use our one sided theorems to obtain improved lower bounds for the absolute values of power sums.

Theorem 3. *Let*

$$B_\nu = \sum_{k=1}^{n} b_k^\nu, \qquad A = B_1^2 - B_2, \qquad \text{and} \qquad B = B_2^2 - B_4.$$

Then one has that

$$\max_{\nu=1,\dots,m} |g(\nu)|^2 \geq B_2 + \frac{B(1+1/m)}{2B_2} - \frac{AB_2 + A^2}{2B_2 m}. \tag{15}$$

One also has that

$$\max_{\nu=1,\dots,m} |g(\nu)|^2 \geq 2B_2 - 2 \times \frac{A^2 - B + mB_4}{B(m+1) - AB_2 - A^2} \tag{16}$$

when $m \geq (B - A^2)/B_4$ and both the numerator and the denominator in the last fraction is positive (this is true for m sufficiently large).

Proof. Let $g(\nu)$ be defined by equation (4). Then

$$\begin{aligned} |g(\nu)|^2 &= \sum_{k=1}^{n} b_k^2 + \sum_{k=1}^{n^2-n} c_k e(\lambda_k \nu), \\ &= B_2 + h(\nu) \end{aligned}$$

where $c_k = b_i b_j$ and $\lambda_k = \theta_i - \theta_j$ for $i \neq j$. It is clear that $h(\nu)$ is real valued and hence we can use the methods of section 2.

Let us now assume that $|h(\nu)| \leq B_2$ for $\nu = 1, \ldots, m$. By using Theorem 1 with $M = B_2$ we have that there exist a ν with $\nu = 1, \ldots, m$ such that

$$g^+(\nu) \geq \frac{B(m+1) - AB_2 - A^2}{2B_2 m}.$$

This implies (15) in case $|h(\nu)| \leq B_2$.

We have by the definition of B that

$$B(m+1) - A^2 - mB_2^2 = B - A^2 - mB_4$$

which is negative if $m \geq (B - A^2)/B_4$, and it follows from Theorem 2 with $M = B_2$ that

$$h(\nu) \geq B_2 - \frac{2A^2 - 2B + 2mB_4}{B(m+1) - AB_2 - A^2}$$

for some $\nu = 1, \ldots, m$. This implies (16) in case $|h(\nu)| \leq B_2$.

Let us now assume that $|h(\nu)| > B_2$ for some $\nu = 1, \ldots, m$. Since $|g(\nu)|^2 = B_2 + h(\nu) \geq 0$ this means that $h(\nu) > B_2$ and $|g(\nu)|^2 \geq 2B_2$. We see that this implies (15) since $2B_2 \geq B_2 + (1 + 1/m)B/(2B_2)$ and the third term on the right hand side in (15) is negative. Likewise it implies (16) since the last term on the right hand side in (16) is negative. □

4 The pure power sum case

In the pure power sum case we have that $B_k = n$, $A = B = n^2 - n$ in Theorem 3 and it follows that

Corollary 1. *One has that*

(i) $$\max_{\nu=1,\ldots,m} |S(\nu)|^2 \geq n + \frac{(-1+n)(1+m-n^2)}{2m},$$

(ii) $$\max_{\nu=1,\ldots,m} |S(\nu)|^2 \geq 2n - \frac{2(1+m-2n^2+n^3)}{(-1+n)(1+m-n^2)}. \qquad (m > n^2)$$

Corollary 1 (i) improves upon known results for m bigger than n^2. In fact it is convenient to write $m = n^2 + j$ and we obtain

Theorem 4. *One has for $j \geq 0$ that*

$$\max_{\nu=1,\ldots,n^2+j} |S(\nu)| \geq \sqrt{n + \frac{(1+j)(n-1)}{2(j+n^2)}}.$$

For $j = 0$ and in the case of unimodular numbers z_k it improves slightly on the general lower bound

$$\max_{\nu=1,\dots,n^2} |S(\nu)| \geq \sqrt{n}$$

in Turán's problem 10 from Andersson [And96]. We obtain

Corollary 2. *One has that*

$$\inf_{|z_k|=1} \max_{\nu=1,\dots,n^2} |S(\nu)| \geq \sqrt{n + \frac{1}{2n} - \frac{1}{2n^2}}.$$

Another result where the lower bound follows from Theorem 4 and the upper bound follows from Montgomery's construction (see Montgomery [Mon94] page 101. Example 6.) is the following

Corollary 3. *Suppose that $n+1$ is a prime number. One then has that*

$$\sqrt{n + \frac{1}{2} - \frac{2n-1}{2(n^2+n-1)}} \leq \inf_{|z_k|=1} \max_{\nu=1,\dots,n^2+n-1} |S(\nu)| \leq \sqrt{n+1}.$$

We remark that the lower bound holds in general. We see that this approximately halves the previous gap between the upper and lower bound. For further discussions of explicit constructions that yields similar upper bounds in power sum problems, see our paper Andersson [Anda].

In our paper Andersson [Andb] page 17, we considered functions Λ that fulfills

$$\sqrt{n}(\Lambda(\alpha) - o(1)) \leq \inf_{|z_k|=1} \max_{\nu=1,\dots,\lfloor \alpha n^2 \rfloor} |S(\nu)|.$$

We proved that we can choose $\Lambda(\alpha) = 1$ for $\alpha > 0$ and furthermore that for $0 < \alpha \leq 1$ we proved that it is the best possible. We asked whether the function must be identically 1 or must be bounded. While we can not answer if there exist such an unbounded function it follows from Corollary 1 that we can choose a Λ such that $\lim_{\alpha \to \infty} \Lambda(\alpha) = \sqrt{2}$. More specifically we obtain the following Theorem.

Theorem 5. *Let $\alpha \geq 1$ be a constant. One then has that*

$$\left(\sqrt{\Phi(\alpha)} - o(1)\right)\sqrt{n} \leq \inf_{|z_k|=1} \max_{\nu=1,\dots,\lfloor \alpha n^2 \rfloor} |S(\nu)| \leq \left(\sqrt{\lceil \alpha \rceil} + o(1)\right)\sqrt{n},$$

where

$$\Phi(\alpha) = \begin{cases} \frac{3}{2} - \frac{1}{2\alpha}, & 1 \leq \alpha \leq 3, \\ 2 - \frac{2}{\alpha}, & \alpha \geq 3. \end{cases}$$

Proof. The lower bound for $1 \leq \alpha \leq 3$ follows from Corollary 1 (i) with $m = \lfloor \alpha n^2 \rfloor$. The lower bound for $3 \leq \alpha$ follows from Corollary 1 (ii) with $m = \lfloor \alpha n^2 \rfloor$. The upper bound follows from Theorem 6 in Andersson [Andb]. $\square$

In particular this will give us

$$\left(\sqrt{\frac{5}{4}} - o(1)\right)\sqrt{n} \leq \inf_{|z_k|=1} \max_{\nu=1,\ldots,2n^2} |S(\nu)| \leq \left(\sqrt{2} + o(1)\right)\sqrt{n}.$$

We see that the lower and upper bounds are not the same and we do not yet have the true asymptotic. This contrasts to the case when we take the maximum over the interval $\nu = 1, \ldots, n^2$ where we proved (see Andersson [Andb])

$$\inf_{|z_k|=1} \max_{\nu=1,\ldots,n^2} |S(\nu)| \sim \sqrt{n}.$$

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